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Variational Iteration Method for Solving Partial Differential Equation Arising from Modeling Heat Transfer in Human Tooth

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Abstract

The paper presents the approximate analytical solution for the partial differential equation (PDE) describing heat transfer in human tooth. The equation governing the phenomenon is solved analytically using variational iteration method (VIM). The results obtained are presented graphically and discussed. It is observed from the results obtained that the temperature was significantly influenced by the parameters.

Keywords: VIM; Human Tooth; PDE; Mathematical Model; Heat Generation.

1. Introduction

Variational iteration method (VIM) is widely used by many researchers to study linear and non-linear partial differential equations. Various problems in Physics, Chemistry, and Biology and Engineering science are modelled mathematically by partial differential equations. In this work we consider He's variational iteration method as a well-known method for finding approximate solutions of partial differential equation arising from mathematical modelling of human tooth.

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This technique was developed by the Chinese mathematician, He [3]. Reference [2] applied Vim to solve nonlinear gas dynamics equation and Stefan equation, Reference [1] used VIM to solve coupled Schrodinger *KdV* equations and shallow water equations. This iterative method has proven rather successful in dealing with mathematical models which give rise to both linear and nonlinear partial differential equations.

Human tooth functions to mechanically breakdown food items by cutting and crushing, this leads to heat transfer which is based on heat conduction process. As hard as our teeth are, they are not immune to extreme hot or cold temperatures. Teeth are porous and sensitive in nature and are used to our normal body temperatures, so when they encounter something hotter or colder while eating and drinking, they can experience issues that may cause great pain or at least mild irritation [8]. Thermal stress is the force that occur when there is a sudden change in temperature and possibly causes graze lines in tooth, human's teeth have ability to hot and cool food and so are subjected to many temperature changes every day [1]. The thermal environment of teeth during daily life varies over a wide range of temperatures (-5 to 76.3^{0}C) [4], hence there is an urgent need to better understand heat transfer process in tooth, thermally induced damage of tooth and the corresponding tooth thermal pain. Human tooth is made up of dentin covered with enamel, the dental expands and contracts slower than the enamel. When a tooth is exposed to a sudden temperature change the junction between the dentin and enamel experiences stress which can result in crack formation [1]. We investigate heat transfer in tooth by mathematical modelling which supplies estimates of information of temperature change, scale thermal conductivity and blood perfusion rate which can be validated, this paper is however limited to the mathematical modelling and simulation of the phenomenon.

2. Model Formulation

Following Pennes' model [7] given as

$$\rho c_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \rho_h c_h \omega_h (T_0 - T) + Q_1 + Q_2 \tag{1}$$

With the initial and boundary conditions formulated as

$$T(x,0) = (T_a - T_0)\frac{x}{L} + T_0,\ T(x,0) = (T_a - T_0)\frac{x}{L} + T_0,\ T(x,0) = (T_a - T_0)\frac{x}{L} + T_0 \tag{2}$$

where $\rho(kgm^{-3})$ is the tissue density, $c(jkg^{-1}k^{-1})$ is the tissue specific heat, $k(wm^{-1}k^{-1})$ is the tissue temperature, $\rho_h(kgm^{-3})$ is the blood density, $c_h(jkg^{-1}k^{-1})$ is the blood specific eat, $T_a(k)$ is the blood temperature, $w_h(m/s^{-1})$ is the blood perfusion rate, $T(k)$ is the temperature of the surrounding tissue, Q_1 is the metabolism heat generation, and Q_2 is heat generation due to external heat sources.

3. Method of solution

3.1 Non-dimensionalization

Here, we non-dimensionalised, (1) and (2) using the following set of dimensionless variables:

$$\boldsymbol{x' = \frac{x}{L}, t' = \frac{t}{t^*}, \theta = \frac{T-T_0}{T_a-T_0}} \tag{3}$$

and we obtained, after dropping prime

$$\boldsymbol{\frac{\partial\theta}{\partial t} = k\frac{\partial}{\partial x}\left(\frac{\partial\theta}{\partial x}\right) + \alpha(1-\theta) + Q_1 + Q_2} \tag{4}$$

$$\boldsymbol{\theta(x,0) = x,\ \theta(0,t) = 0,\ \theta(L,t) = 0} \tag{5}$$

3.2 Solution by Variational Iteration Method

Here, we solve equations (4) and (5) using variational iteration method. We consider the following general nonlinear system:

$$\boldsymbol{L\theta(x,t) + N\theta(x,t) = g(x,t)} \tag{6}$$

where $\boldsymbol{L}$ is a linear operator and $\boldsymbol{N}$ is a nonlinear operator, and $\boldsymbol{g(x,t)}$ is the inhomogenous term. In the variational iteration method, a correctional function for (6) can be written as:

$$\boldsymbol{\theta_{n+1}(x,t) = \theta_n(x,t) + \int_0^t \lambda(L\theta(x,s) + N\theta(x,s) - g(x,s)\,)\,ds} \tag{7}$$

The successive approximation $\boldsymbol{\theta_n, n \geq 0}$ can be established by determining $\boldsymbol{\lambda}$, a general Lagrange multiplier, which can be identified optimally via the variational theory [6], the function $\boldsymbol{\widetilde{\theta_n}}$ is a restricted variation, which means $\boldsymbol{\delta\widetilde{\theta_n} = 0}$.

Now, we get the correction function

$$\boldsymbol{\theta_{n+1}(x,t) = \theta_n(x,t) + \int_0^x \lambda(s)\left(-\frac{\partial^2}{\partial x^2}\theta(s,t) + N\widetilde{\theta}(s,t) + \alpha\theta(s,t) - \alpha - Q_1 - Q_2\right)ds} \tag{8}$$

where $\boldsymbol{N\widetilde{\theta}(s,t) = \frac{\partial}{\partial t}\theta(x,t)}$, taking variation with respect to the independent variable $\boldsymbol{\theta_n}$ noticing that

$\boldsymbol{\delta N\widetilde{\theta}(s,t) = 0}$.

$$\boldsymbol{\delta\theta_{n+1}(x,t) = \delta\theta_n(x,t) + \delta\int_0^x \lambda(s)\left(-\frac{\partial^2}{\partial x^2}\theta(s,t) + N\widetilde{\theta}(s,t) + \alpha\theta(s,t) - \alpha - Q_1 - Q_2\right)ds} \tag{9}$$

This yields the stationary conditions

$$\left.\begin{aligned} \boldsymbol{1 - k\lambda(s)} &\boldsymbol{= 0} \\ \boldsymbol{k\lambda} &\boldsymbol{= 0} \\ \boldsymbol{k\lambda'' + \alpha\lambda} &\boldsymbol{= 0} \end{aligned}\right\} \tag{10}$$

This in turn gives $\boldsymbol{\lambda = \sin\left(\sqrt{\frac{\alpha}{k}}(s-x)\right)}$, substituting this value of the Lagrange multiplier into the functional

(9) gives the iteration formula

$$\left.\boldsymbol{\theta_{n+1}(x,t) = \theta_n(x,t) + \int_0^x \sin\left(\sqrt{\frac{\alpha}{k}}(s-x)\right)\left(-\frac{\partial^2}{\partial x^2}\theta(s,t) + \alpha\theta(s,t) - \alpha - Q_1 - Q_2\right)ds,\ n > 0}\right\} \quad (11)$$

thus, we can obtain approximation solutions for $\boldsymbol{\theta(x,t)}$, considering the boundary conditions and insert $\boldsymbol{\theta(x,t) \cong \theta_n(x,t)}$.

To solve (4) and (5), we use iterative formula (11) and start with an initial approximation

$$\boldsymbol{\theta_0(x,t) = x} \quad (12)$$

and we obtain the following successive approximations as follows:

$$\theta_1(x,t) = x - \frac{k}{\alpha^2}\begin{pmatrix} Q_2 cos\left(\sqrt{\frac{\alpha}{k}}x\right)\sqrt{\frac{\alpha}{k}} - \alpha sin\left(\sqrt{\frac{\alpha}{k}}x\right) + \alpha cos\left(\sqrt{\frac{\alpha}{k}}x\right)\sqrt{\frac{\alpha}{k}} + \\ Q_1 cos\left(\sqrt{\frac{\alpha}{k}}x\right)\sqrt{\frac{\alpha}{k}} - \alpha\sqrt{\frac{\alpha}{k}} + \alpha x\sqrt{\frac{\alpha}{k}} - Q_2\sqrt{\frac{\alpha}{k}} - Q_1\sqrt{\frac{\alpha}{k}} \end{pmatrix} \quad (13)$$

$$\boldsymbol{Q_2(x,t) = x - \frac{k}{\alpha^2}\begin{pmatrix} -2\alpha\sqrt{\frac{\alpha}{k} - 2Q_2}\sqrt{\frac{\alpha}{k}} - 2Q_1\sqrt{\frac{\alpha}{k}} + \alpha + Q_1 + Q_2 - Q_2 cos\left(\sqrt{\frac{\alpha}{k}}x\right) - \alpha cos\left(\sqrt{\frac{\alpha}{k}}x\right) \\ -Q_1 cos\left(\sqrt{\frac{\alpha}{k}}x\right) - 2\alpha sin\left(\sqrt{\frac{\alpha}{k}}x\right) + 2Q_2 cos\left(\sqrt{\frac{\alpha}{k}}x\right)\sqrt{\frac{\alpha}{k}} + 2\alpha cos\left(\sqrt{\frac{\alpha}{k}}x\right)\sqrt{\frac{\alpha}{k}} + \\ 2Q_1 cos\left(\sqrt{\frac{\alpha}{k}}x\right)\sqrt{\frac{\alpha}{k}} - x\alpha + 2x\sqrt{\frac{\alpha}{k}} - x\alpha cos\left(\sqrt{\frac{\alpha}{k}}x\right) - Q_2 sin\left(\sqrt{\frac{\alpha}{k}}x\right)\sqrt{\frac{\alpha}{k}}x - \\ Q_1 sin\left(\sqrt{\frac{\alpha}{k}}x\right)\sqrt{\frac{\alpha}{k}}x - \alpha sin\left(\sqrt{\frac{\alpha}{k}}x\right)\sqrt{\frac{\alpha}{k}}x + 2k sin\left(\sqrt{\frac{\alpha}{k}}x\right)\sqrt{\frac{\alpha}{k}} \end{pmatrix}} \quad (14)$$

The computations were done using computer symbolic algebraic package MAPLE 2016 [5].

4. Results

We solve the partial differential equation describing the heat transfer on tooth using variational iteration method. Numerical solutions of equation (14) are computed for the values of $\boldsymbol{\alpha = 0.2}$, $\boldsymbol{k = 0.3, 0.5, 0.7}$, $\boldsymbol{Q_1 = 0.2, 0.4, 0.6}$, and $\boldsymbol{Q_2 = 0.3, 0.5, 0.7}$.

The following figures explain the temperature distribution in tooth.

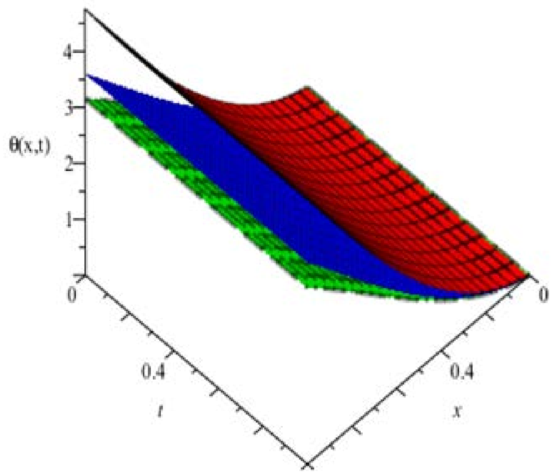

Figure 1: Variation of temperature, Θ(x,t) with blood perfusion rate, α

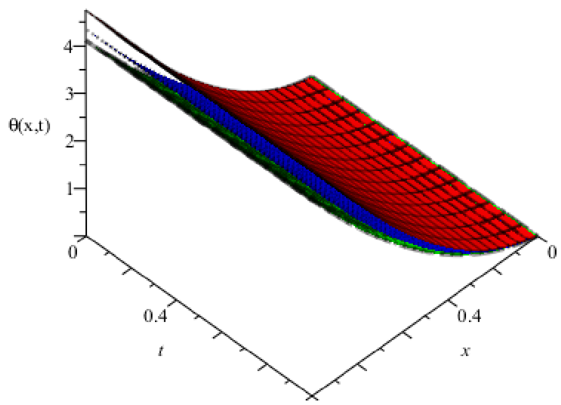

Figure 2: Variation of temperature, Θ(x,t) with scale therm al conductivity, k

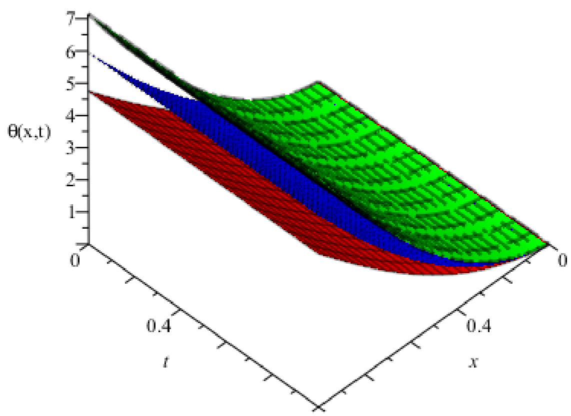

Figure 3: Variation of temperature, Θ(x,t) with metabolism heat generation, Q1

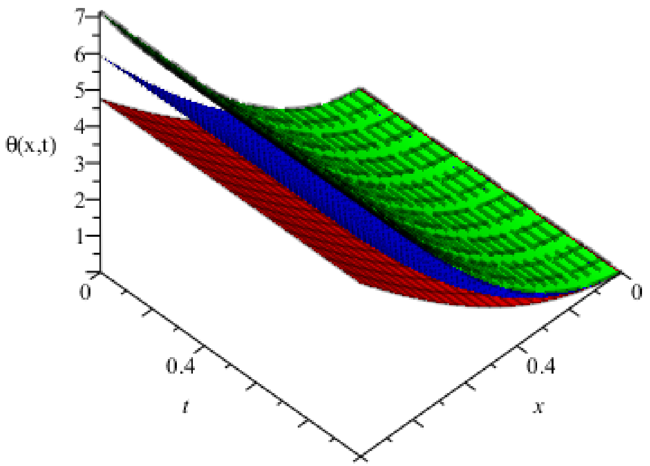

Figure 4: Variation of temperature, Θ(x,t) with heat generation due to external sources, Q2

5. Discussion

Figure 1 shows the variation of temperature, $\theta(x,t)$ with blood perfusion rate, α. It is observed that the temperature decreases along time, t but increases along distance, x. The temperature also decreases as α increases.

Figure 2 shows the variation of temperature, $\theta(x,t)$ with scale thermal conductivity, k. It is observed that the temperature decreases along time, t but increases along distance, x. The temperature also decreases as k increases.

Figure 3 shows the variation of temperature, $\theta(x,t)$ with metabolism heat generation, Q_1. It is observed that the temperature decreases along time, t but increases along distance, x. The temperature also increases as Q_1 increases.

Figure 4 shows the variation of temperature, $\theta(x,t)$ with heat generation due to external sources, Q_2. It is observed that the temperature decreases along time, t but increases along distance, x. The temperature also increases as Q_2 increases.

Dental procedures lead to increase in temperature of the tooth and supportive tissue, hence heat source activation should not be prolonged to protect the pulp from heat damage.

6. Conclusion

A mathematical model based on heat transfer in human tooth is presented. Variational iteration method has been used to obtain the approximate solution of the model putting into consideration blood perfusion rate, scale thermal conductivity, metabolism heat generation and heat due to external sources as the governing parameters. It was discovered that the temperature was significantly influenced by the parameters.

7. Recommendations

This work can be extended by researchers, finding the uniqueness and existence of the model can yield better results. The effect of other parameters on the system response can also be looked into for further studies.

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